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# TRANSFORMATIONS OF THE JACOBIAN AMPLITUDE FUNCTION AND ITS CALCULATION VIA THE ARITHMETIC-GEOMETRIC MEAN*

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**Abstract.** With the aid of the Poisson summation formula, expressions for the Jacobian amplitude function, am $(z; m)$, along with the complete set of Jacobian elliptic functions are given that, aside from their branchpoints and poles, respectively, are convergent throughout the complex plane for arbitrary parameter $m$. By utilizing the expression for am $(z; m)$, its periodicity properties are determined in each of the regions $m<0, 0<m<1$, and $m>1$. Novel yet fundamental identities are presented describing various linear and quadratic transformations of the Jacobian amplitude function. Finally, that method based on the arithmetic-geometric mean and most widely employed for calculating the Jacobian elliptic functions is shown to be, when interpreted explicitly in terms of am $(z; m)$ and its transformation properties, a method first and foremost for the calculation of the Jacobian amplitude and co-amplitude functions from which the elliptic functions themselves are subsequently evaluated by means of simple, trigonometric identities.

**Key words.** Jacobian amplitude function transformations, Jacobian elliptic functions, arithmetic-geometric mean

**AMS(MOS) subject classifications.** 33A25, 30D99, 41A58

**1. Introduction.** The most familiar of the twelve-member family of Jacobian elliptic functions (JEF) is the copolar trio

$$
\begin{aligned}
sn(z;m)&=\sin[\mathrm{am}\,(z;m)]=\frac{\Theta_3}{\Theta_2}\frac{\Theta_1(z/\Theta_3^2;q)}{\Theta_4(z/\Theta_3^2;q)},\\
cn(z;m)&=\cos[\mathrm{am}\,(z;m)]=\frac{\Theta_4}{\Theta_2}\frac{\Theta_2(z/\Theta_3^2;q)}{\Theta_4(z/\Theta_3^2;q)},\\
dn(z;m)&=\frac{d}{dz}\mathrm{am}\,(z;m)=\frac{\Theta_4}{\Theta_3}\frac{\Theta_3(z/\Theta_3^2;q)}{\Theta_4(z/\Theta_3^2;q)},
\end{aligned}
\tag{1.1}
$$

where am $(z; m)$ is the Jacobian amplitude function, $m$ is the Jacobian parameter ($k=+m^{1/2}$ is the modulus), $q=\exp[-\pi K'(m)/K(m)]$ is the nome with $K(m)$ and $K'(m)=K(1-m)$ the Jacobian quarter periods, and $\Theta_i(z;q), i=1,\cdots,4$, are the theta functions with $\Theta_i$ denoting $\Theta_i(z=0;q)$. The remaining members of the JEF family can be defined directly either as reciprocals or ratios of these three functions or by adding to the argument $z$ one or both of the quarter periods, e.g., $cd(z;m)=cn(z;m)/dn(z;m)=sn(z+K;m)$. In what follows we will assume that the parameter $m$ is real but otherwise arbitrary while the variable $z=x+iy$ is, in general, arbitrary and complex. Comprehensive descriptions of elliptic functions and JEF in particular may be found in [8], [10], [20], and [23], while extensive compendia of the properties of JEF are given in [5], [13], [15], and [17]. In general, well-known identities involving JEF will be cited without specific reference since they may be found in any of the aforementioned works.

The canonical definitions of the JEF given by (1.1) represent two characteristically distinct approaches to the description of these functions. Historically, the JEF were first defined as inverses of elliptic integrals with the basis of this approach summarized

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by the fundamental identity am $(z; m) = F^{-1}(z; m)$ where $F(z; m)$ is the elliptic integral of the first kind (a historical account of the development of elliptic function theory is given by Alling [1]). However, the study of the JEF via the amplitude function is not and has never been the favored approach principally for the reason that, since am $(z; m)$ is not itself an elliptic function, this approach could not effectively exploit the many general and powerful theorems for elliptic functions but would instead be forced to rely almost exclusively on "brute force" algebraic methods. With origins traceable to Jacobi's seminal work *Fundamenta Nova* [12], the preferred approach to the study of the JEF has been through the theta functions, which, of course, are entire functions with simple zeros. In modern texts on elliptic function theory (e.g., Chandrasekharan [8]), the function am $(z; m)$ is ignored altogether.

To describe the amplitude function thus as one of the more obscure higher transcendental functions would be an understatement. The extent of its inconspicuousness is best illustrated with the example of the classical problem of the simple pendulum. The angular displacement $\Theta$ of a point mass $\mu$ constrained to swing in a vertical plane by a massless, rigid rod of length $R$ is described by the nonlinear equation (Whittaker [22])

$$\frac{d^2\Theta}{dt^2}+\frac{g}{R}\sin\Theta=0, \tag{1.2}$$

or, equivalently,

$$\frac{1}{2}\mu R^2\left[\frac{d\Theta}{dt}\right]^2+\mu gR(1-\cos\Theta)=E_0 \tag{1.3}$$

where the total energy $E_0$ is a constant. Note that (1.2) is also identical in form to the traveling wave, sine-Gordon equation. With the most general possible initial conditions of $\Theta=0$ and $d\Theta/dt=[2E_0/\mu R^2]^{1/2}$ for $t=0$ (this choice places no restrictions on the value of $E_0$), the exact general solution to (1.2) and (1.3) is simply

$$\Theta(t)=2\cdot\text{am}\left[\left(\frac{g}{mR}\right)^{1/2}t;\, m\right],\qquad m=\frac{2\mu gR}{E_0} \tag{1.4}$$

a result that follows immediately from the identities $(d/dx)\text{am}(x; m)=dn\,(x; m)$ and $(d/dx)\,dn\,(x; m)=-(m/2)\sin[2\text{am}\,(x; m)]$. Despite the simplicity of this result, the explicit solution (1.4) has heretofore never been published even though dozens of texts and papers have treated the simple pendulum problem "exactly." Invariably, these "exact" treatments solve not explicitly for $\Theta(t)$ but rather for the variable $\sin(\Theta/2)$ (see, e.g., Whittaker [22] and Alling [1]) and, furthermore, choose to either ignore entirely the rotating $(m<1)$ pendulum by adopting initial conditions that restrict the value of $m$ to $m>1$, or to treat the cases of $m<1$ and $m>1$ as distinct problems (the special case of $m=1$ is also often treated separately). The distinction, however, between $\Theta(t)$ and $\sin(\Theta/2)$ is not a trivial one; the latter is a true doubly periodic function for all values of the parameter $m\neq 1$ whereas the amplitude function possesses a real period if and only if $m>1$, i.e., only the am $(z; m)$ solution as given in (1.4) explicitly and unequivocally distinguishes between the oscillating $(m>1)$ and rotating $(m<1)$ pendulum solutions. In addition, it is important to note that (1.4) is a solution to the pendulum equation (1.2) for arbitrary values of $m$, i.e., it is solely the initial conditions that determine the specific value of the parameter $m$. Thus we have, from (1.4), that $\sin(\Theta/2)=sn[(g/mR)^{1/2}t;\, m]=k^{-1}sn[(g/R)^{1/2}t;\, 1/m]$, revealing that both cases of a parameter greater than 1 and less than 1 (as well as $m=1$) are succinctly and completely delineated by the result in (1.4) and that, in contrast, the "traditional" division of this problem into two (or three) distinct cases is unnecessarily redundant.

In the following, utilizing a completely novel representation for am $(z; m)$ that is convergent throughout the complex plane excepting only the logarithmic branchpoints of the function (hereafter the term "unrestricted representation" will be used to denote any representation of a function that is valid throughout the complex plane except at any isolated, singular points and/or branchpoints of the function), we examine its periodicity for all real values of $m$. We also present various linear and quadratic transformations of the amplitude function corresponding to, e.g., the complementary parameter transformation, the Landen and Gauss transformations, etc. Although the expression of these transformations in terms of the JEF are well known, the results presented here for am $(z; m)$ are, with one exception, new results. As will be evident, the transformations for am $(z; m)$ offer concise, straightforward representations for these transformations and, in certain cases, offer a simple representation for which the corresponding JEF transformation is considerably more complicated. An example of the latter is the ascending Landen transformation that takes a simple form for am $(z; m)$ whereas the identities involving the JEF are algebraic. In addition, the formulae presented here offer further insight into the nature of these basic transformations beyond that associated strictly with the JEF formulae.

Principally for reasons of computational efficiency, the most widely used method for calculating the JEF (and elliptic integrals) is that based on the arithmetic-geometric mean along with various supplemental relations normally involving specific transformations directly related to the function to be evaluated (see the general articles by King [14], Carlson [6], and Milne-Thomson [17]). The term "arithmetic-geometric mean" will henceforth be understood to include whatever supplemental relations are used in conjunction with the arithmetic-geometric mean itself in the overall calculation of the specific function in question. The final section of this paper describes the method of the arithmetic-geometric mean explicitly in terms of the amplitude function and its transformation properties and will demonstrate that the method of the arithmetic-geometric mean is first and foremost a technique for the calculation **NOT** of the JEF but rather of am $(z; m)$ directly (along with the "coam" function am $(K-z; m)$). It is emphasized that the intent of this section is not to define or present algorithms for the arithmetic-geometric mean as applied to the calculation of the JEF; there exist several excellent, comprehensive descriptions of this technique [1], [6], [7], [9], [14], [16], including strictly algebraic versions [6], [21], computer algorithms [4], [11], as well as versions permitting complex parameters [9]. Rather, we wish to show that the actual basis for this technique is best and most clearly described in terms of the transformation formulae for am $(z; m)$ presented in the first parts of this paper.

**2. Unrestricted representations for the Jacobian functions.** The Fourier series for the functions $dn(z; m)$ and am $(z; m)$ may be written in the following form:

$$dn(z;m)=\frac{\pi}{2K}+\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos\left(\frac{n\pi z}{K}\right) = \frac{\pi}{2K}\sum_{n=-\infty}^{\infty}\operatorname{sech}\left[n\pi\frac{K'}{K}\right]e^{in\pi z/K} \tag{2.1}$$

and

$$\operatorname{am}(z;m)=\int_0^z dn(z;m)\,dz=\frac{\pi z}{2K}+2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sin\left(\frac{n\pi z}{K}\right). \tag{2.2}$$

However, as a consequence of the fact that the variable $z$ and the summation index $n$ are cofactors in the Fourier series representations, these expressions are only valid throughout the restricted domain $|\mathrm{Im}\,(z/K)| < \mathrm{Im}\,(iK'/K)$. For example, for $0 < m < 1$ where both $K$ and $K'$ are real, these expressions are valid only in the infinite strip $|\mathrm{Im}\,(z)| < K'(m)$. In addition, or rather as a result of this limitation, the Fourier series such as that given by (2.1) for $dn(z; m)$ account explicitly only for the periodicity properties with respect to the quarter-period $K(m)$ and completely fail to describe the behavior with respect to the quarter-period $iK'(m)$. To arrive at an unrestricted representation for $dn(z; m)$, we apply the Poisson summation formula (see Bellman [2] for a discussion and examples of the applicability of this formula)

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, e^{2\pi i n u}\, du\right] \tag{2.3}$$

to the second of (2.1) with $f(n) = \mathrm{sech}\,[n\pi K'/K] \exp\{in\pi z/K\}$. Replacing the variable "$n$" with "$u$" and evaluating the integral given in (2.3) leads directly to the result

$$dn(z; m) = \frac{\pi}{2K'} \sum_{n=-\infty}^{\infty} \mathrm{sech}\left[\frac{\pi K}{K'}\left(n + \frac{z}{2K}\right)\right]. \tag{2.4}$$

This expression for the function $dn(z; m)$ is superior to the Fourier series representation (2.1) in that, (a) equation (2.4) is convergent throughout the entire complex plane, poles excepted, and, (b) partly as a consequence of this, it describes equally explicitly the periodicity of $dn(z; m)$ with respect to *both* $K(m)$ and $K'(m)$ (the actual periods are $2K(m)$ and $4iK'(m)$). Indeed, since the variable $z$ and the summation index $n$ appear as additive terms in (2.4) in contrast to the Fourier series, (2.1), where they are multiplicative factors, the *only* condition required to ensure convergence of the expression (2.4) is $\mathrm{Re}\,(K/K') \neq 0$ which, for real $m$, is equivalent to $m \neq 0$.

The analogous Poisson-sum-transformed expressions for $sn(z; m)$ and $cn(z; m)$, from which the remaining members of the JEF family are derived as noted previously, are found by following exactly similar procedures as for the case of $dn(z; m)$ above, i.e., the Fourier series for these functions are first converted to a summation over an index "$n$" running from $-\infty$ to $+\infty$, the summation term is substituted into (2.3), and, following a substitution of the variable "$n$," the integration is performed. The final results may be compactly expressed in the following form, with $A = \pi/2K'$:

$$\begin{aligned} dn(z; m) &= A \sum_{n=-\infty}^{\infty} \mathrm{sech}\left[\frac{\pi K}{K'}\left(n + \frac{z}{2K}\right)\right], \\ k \cdot cn(z; m) &= A \sum_{n=-\infty}^{\infty} (-1)^n\, \mathrm{sech}\left[\frac{\pi K}{K'}\left(n + \frac{z}{2K}\right)\right], \\ k \cdot sn(z; m) &= A \sum_{n=-\infty}^{\infty} (-1)^n \tanh\left[\frac{\pi K}{K'}\left(n + \frac{z}{2K}\right)\right], \end{aligned} \tag{2.5}$$

$$\begin{aligned} k' \cdot nd(z; m) &= A \sum_{n=-\infty}^{\infty} \mathrm{sech}\left[\frac{\pi K}{K'}\left(n + \frac{1}{2} + \frac{z}{2K}\right)\right], \\ -kk' \cdot sd(z; m) &= A \sum_{n=-\infty}^{\infty} (-1)^n\, \mathrm{sech}\left[\frac{\pi K}{K'}\left(n + \frac{1}{2} + \frac{z}{2K}\right)\right], \\ k \cdot cd(z; m) &= A \sum_{n=-\infty}^{\infty} (-1)^n \tanh\left[\frac{\pi K}{K'}\left(n + \frac{1}{2} + \frac{z}{2K}\right)\right], \end{aligned} \tag{2.6}$$

$$cs(z;m)=A\sum_{n=-\infty}^{\infty}\operatorname{csch}\left[\frac{\pi K}{K'}\left(n+\frac{z}{2K}\right)\right],$$

$$ds(z;m)=A\sum_{n=-\infty}^{\infty}(-1)^n\operatorname{csch}\left[\frac{\pi K}{K'}\left(n+\frac{z}{2K}\right)\right], \tag{2.7}$$

$$ns(z;m)=A\sum_{n=-\infty}^{\infty}(-1)^n\coth\left[\frac{\pi K}{K'}\left(n+\frac{z}{2K}\right)\right],$$

$$-k'\cdot sc(z;m)=A\sum_{n=-\infty}^{\infty}\operatorname{csch}\left[\frac{\pi K}{K'}\left(n+\frac{1}{2}+\frac{z}{2K}\right)\right],$$

$$k'\cdot nc(z;m)=A\sum_{n=-\infty}^{\infty}(-1)^n\operatorname{csch}\left[\frac{\pi K}{K'}\left(n+\frac{1}{2}+\frac{z}{2K}\right)\right], \tag{2.8}$$

$$dc(z;m)=A\sum_{n=-\infty}^{\infty}(-1)^n\coth\left[\frac{\pi K}{K'}\left(n+\frac{1}{2}+\frac{z}{2K}\right)\right].$$

The symmetry of these expressions is striking; with the exception of certain factors of $\pm i$, the right-hand sides of (2.5)-(2.8) are, interestingly, exactly the set of (symmetrical) primitive elliptic functions originally defined by Neville [18], [19]. All twelve of these expressions are valid throughout the complex plane for arbitrary $m \neq 0$, their respective poles excepted. Each of the numbered equations represents a copolar trio of the JEF while the three quartets formed from the respective members of each of these trios are coperiodic. The expressions for *sn*, *cn*, and *dn* recently have been presented and discussed by Boyd [3]. However, to the best of the author's knowledge, (2.5)-(2.8) for the complete JEF family have not been published previously.

Integration of (2.4) results in an expression for the amplitude function in the form

$$\operatorname{am}(z;m)=\sum_{n=-\infty}^{\infty}\operatorname{gd}\left[\frac{\pi K}{K'}\left(n+\frac{z}{2K}\right)\right] \tag{2.9}$$

$$=\operatorname{gd}\left[\frac{\pi z}{2K'}\right]+\sum_{n=1}^{\infty}\left[\operatorname{gd}\left[\frac{\pi K}{K'}\left(n+\frac{z}{2K}\right)\right]-\operatorname{gd}\left[\frac{\pi K}{K'}\left(n-\frac{z}{2K}\right)\right]\right] \tag{2.10}$$

where $\operatorname{gd}(z)$ is the Gudermannian function. Equations (2.9) and (2.10) converge throughout the complex plane except at the logarithmic branchpoints $z = 2sK+(2t+1)iK'$ where $s$ and $t$ are arbitrary integers. Equation (2.9) will serve as the basis for the derivation of the various identities in the following work so that the results obtained will be valid without restrictions on the range of $z$; those results obtained by direct reference to the analogous JEF relations, i.e., by "inversion," are generally accompanied by restrictions on the range of the real and/or imaginary parts of the variable $z$.

It is worthwhile noting that although the unrestricted representations given above for the JEF and for $\operatorname{am}(z;m)$ are much more attractive for analytical purposes than their limited Fourier series counterparts, neither set of expressions, for purposes of numerical calculation, is as computationally efficient as those methods based either on the arithmetic-geometric mean or on the use of the theta functions (cf. [4], [6], [11], [14]). High precision, numerical evaluation of the JEF or $\operatorname{am}(z;m)$ using the Fourier series is truly practical only when $m \approx 0$ or, using the expressions above, when $m \approx 1$.

Since the characteristics of the function $\operatorname{am}(z;m)$ in the complex plane are so intimately connected with the nature of the Gudermannian function, a brief accounting of gd $(z)$ is in order at this point. Because the function sech $(z)$ is a singly periodic, meromorphic function with simple poles at the points $(2t+1)i\pi/2$ with residues of $(-1)^{t+1}i$ where $t=0, \pm1, \pm2, \cdots$, the Gudermannian function defined as the definite integral (Jahnke and Emde [13])

$$\text{gd}(z)=\alpha+i\beta=\int_0^z \text{sech}(z)\,dz, \qquad z\neq(2n+1)\frac{i\pi}{2},$$

where $\alpha$ and $\beta$ are strictly real, is a single-valued, analytic function provided that the complex plane is cut along the branchlines lying on the imaginary axis from $(4t+1)i\pi/2$ to $(4t+3)i\pi/2$ (logarithmic branchpoints for gd $(z)$) where $t$ is an arbitrary integer. Relations such as $\sinh(z)=\tan[\text{gd}(z)]$ and $\cosh(z)=\sec[\text{gd}(z)]$ follow directly from the definition above. The real and imaginary parts of gd $(z)$ for $x\neq0$ are given explicitly and uniquely by the relations

$$\begin{aligned}\alpha&=\text{gd}(x)+\tan^{-1}[\text{csch}(x)]-\tan^{-1}[\cos(y)\,\text{csch}(x)],\\ \beta&=\tanh^{-1}[\sin(y)\,\text{sech}(x)]\end{aligned} \tag{2.11}$$

with $\text{gd}(x)=2\tan^{-1}[\tanh(x/2)]$ and where $|\alpha|<\pi$ and $|\alpha|\to\pi/2$, $|\beta|\to0$ as $|x|\to\infty$ for all $y$. For $x=0$ and $y\neq(2t+1)i\pi/2$, we have $\alpha=0$ and $\beta=\tanh^{-1}[\sin(y)]$. Note that gd $(z)$ is singly periodic with period $2\pi i$ and that $\text{gd}(-z)=-\text{gd}(z)$ and $\text{gd}(z^*)=\text{gd}^*(z)$. Expanding sech $(z)$ in terms of $\exp(\pm z)$ and integrating leads to an unrestricted representation for the Gudermannian function in the form

$$\text{gd}(z)=\text{sgn}(x)\frac{\pi}{2}-2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}[\text{sgn}(x)\cosh[(2n+1)z]-\sinh[(2n+1)z]], \tag{2.12}$$

which is convergent throughout the complex plane, the logarithmic branchpoints $z=(2t+1)i\pi/2$ excepted, and where $\text{sgn}(x)=+1, 0$, or $-1$ according to $x>0$, $x=0$, or $x<0$, respectively (the real part of gd $(z)$ vanishes along the imaginary axis). It follows directly from (2.12) that

$$\text{gd}(x+iy\pm in\pi)=\text{sgn}(x)\frac{1-(-1)^n}{2}\pi+(-1)^n\,\text{gd}(x+iy) \tag{2.13}$$

revealing a finite discontinuity (of $2\pi$) in the real part of gd $(z)$ across each of the branchcuts $(x=0, \cos(y)<0)$.

Before proceeding to examine the specific properties of the Jacobian amplitude function, it is appropriate at this point to discuss briefly its general characteristics in the complex plane given the basic results immediately above. Neville [18, pp. 18–20] has shown that the integral of an elliptic function having zero residues defines a doubly (additive) pseudoperiodic, meromorphic function. The function am $(z; m)$, in contrast, as the integral of the elliptic function $dn(z; m)$ having nonzero residues (specifically, $dn(z; m)$ has simple poles with residues of $-i$ at the points $2sK+(4t+1)iK'$ and $+i$ at the poles $2sK+(4t-1)iK'$, where $s$, $t$ are arbitrary integers), is, in general, a doubly pseudoperiodic function with logarithmic branchpoints. Thus am $(z; m)$, were it defined solely by (1.1) and (2.2), would be an infinitely multiple-valued function of $z$ with branches differing by integral multiples of $2\pi$ corresponding to the infinite number of possible paths of integration from zero to $z$ encircling the poles of $dn(z; m)$ in different ways. However, by cutting the complex plane along the line segments joining these logarithmic branchpoints, specifically from $2sK+(4t+1)iK'$ to $2sK+(4t+3)iK'$, where $s$, $t$ are integers, the function am $(z; m)$ is made single-valued and analytic throughout the cut, complex plane. Finally, a principal branch is selected from among these single-valued branches by the requirement am $(z=0; m)=0$. The real part of am $(z; m)$ will be discontinuous (by $2\pi$) across each of the branchlines. In nearly all respects, as the form of (2.9) intimates, the Jacobian amplitude function am $(z; m)$ may be effectively considered as a doubly pseudoperiodic generalization of the singly periodic Gudermannian function gd $(z)$, noting in particular the degeneracy am $(z; m=1)=$ gd $(z)$.

**3. The periodicity properties of am $(z; m)$ for real $m$.** Exactly as with the JEF, the behavior of the function am $(z; m)$ is characteristically different in the three distinct regions: $-\infty<m<0, 0<m<1$, and $1<m<\infty$. From (2.9), which is valid for all parameter values, it follows that the amplitude function is always at least singly periodic when $m\neq 0$, with a period of $4iK'(m)$. Exploiting the fact that the amplitude function for real $m$, like the JEF, is strictly real whenever $z$ is real, expressions for am are given in the regions $m<0$ and $m>1$ that reflect this characteristic. Unless specifically noted, the degenerate cases of am $(z; m=0)=z$ and am $(z; m=1)=$ gd $(z)$ are generally excluded from the relations below. Finally, it should be noted that, in all of the work to follow, the numerical value of the parameter $m$ will generally be restricted to $0<m<1$ and parameters that are less than zero or greater than 1 will then be expressed explicitly in terms of $m$, e.g., a parameter greater than 1 will be represented by $1/m$ where $0<m<1$.

**3.1. am $(z; m)$ for $0<m<1$.** An arbitrary point in the complex plane may be represented as

$$z+2sK+2itK'=x+iy+2sK+2itK', \qquad |x/2K|<1 \quad \text{and} \quad |y/K'|<1 \tag{3.1}$$

where $s$ and $t$ are integers and where, for $0<m<1$, both $K$ and $K'$ are real. Only the lines $y=(2t+1)K'$, which include the logarithmic branchpoints of am $(z; m)$, are excluded from the representation (3.1). When we use (2.12) and the expression for am $(z; m)$ given by (2.10), it is a straightforward task to derive the result:

$$\text{am}\,(z+2sK+2itK'; m)=s\pi+\text{sgn}\,(x)\frac{1-(-1)^t}{2}\pi+(-1)^t\,\text{am}\,(z; m). \tag{3.2}$$

Thus we have am $(z+2sK+2itK'; m)=$ am $(z; m)$ if and only if $s=0$ and $t$ is even and so, for $0<m\leqq 1$, the function am $(z; m)$ is a singly periodic function with the strictly imaginary period $4iK'$. Equation (3.2) corresponds exactly with the relief figures for am $(z; m)$ given by Jahnke and Emde [13]. Note that the branchlines so clearly illustrated in those figures are also explicitly accounted for by (3.2). For example, taking $s=2$, $t=1$, and $z=x>0$, we have am $(4K+2iK'+x; m)=3\pi-$ am $(x; m)$, whereas am $(4K+2iK'-x; m)=\pi+$ am $(x; m)$.

**3.2. am $(z; m)$ for $m<0$.** Denoting a negative parameter (imaginary modulus) by the expression $-m/m'$ where $m'=1-m$ and $0<m<1$, we have the identities [5], [10], [15], [23]

$$K(-m/m')=k'K(m), \qquad K'(-m/m')=k'K'(m)+ik'K(m). \tag{3.3}$$

Note that the sign used in this identity for $K'(-m/m')$ is ambiguous for real $m$; either a + or − may be used (consistently) without affecting the validity of the final results [19, pp. 103-107]. The expression (2.9) for the amplitude function with negative parameter then takes the following form:

$$\text{am}\left(z; -\frac{m}{m'}\right)=\sum_{n=-\infty}^{\infty}\text{gd}\left[\frac{\pi K}{K'+iK}\left(n+\frac{z}{2k'K}\right)\right]. \tag{3.4}$$

This expression is cumbersome in that the individual terms in the summation are complex when $z$ is strictly real even though the function $\operatorname{am}(z; -m/m')$ itself is real in such a case. To overcome this shortcoming, consider the expression

$$G(z)=\sum_{n=-\infty}^{\infty}\left[(1+ia)\operatorname{sech}\left[a\pi\left(z+n+\frac{1}{2}\right)\right]-\operatorname{sech}\left[\frac{a\pi}{1+ia}(z+n)\right]\right]$$

where $a=K/K'$. This function $G(z)$ is a doubly periodic function since $G(z+1)=G(z+2i/a)=G(z)$ with possible poles at the isolated points $z_{st}=(s+\frac{1}{2})+(2t+1)i/2a$, i.e., $G(z)$ is an elliptic function. It is, however, straightforward to prove that the limit $(z-z_{st})\cdot G(z)=0$ as $z\to z_{st}$ so that the function $G(z)$ is indeed without poles. As a consequence of Liouville's theorem, an elliptic function without poles must be a constant and, in fact, we have that $G(z)\equiv 0$. By integration we arrive at the desired result

$$\operatorname{am}\left(z; -\frac{m}{m'}\right)=\sum_{n=-\infty}^{\infty}\operatorname{gd}\left[\frac{\pi K}{K'}\left(n+\frac{1}{2}+\frac{z}{2k'K}\right)\right]-\frac{\pi}{2} \tag{3.5}$$

for which the only complex dependence is implicitly through $z$. Representing an arbitrary point in the complex plane as above with $|x/2k'K|$ and $|y/k'K'|<1$ and using the representation (2.12) in (3.5) leads to the result

$$\begin{aligned}\operatorname{am}&\left[z\pm k'K+2sK\left(-\frac{m}{m'}\right)+2itK'\left(-\frac{m}{m'}\right); -\frac{m}{m'}\right]\\ &=(s-t)\pi+(1\pm\operatorname{sgn}(x))\frac{1-(-1)^t}{2}\pi\\ &\quad+(-1)^t\operatorname{am}\left(z\pm k'K; -\frac{m}{m'}\right).\end{aligned} \tag{3.6}$$

The left-hand side of (3.6) will equal $\operatorname{am}(z\pm k'K; -m/m')$ if and only if $s=t=2L$ so that $\operatorname{am}(z; -m/m')$ is singly periodic with the strictly imaginary period $4K(-m/m')+4iK'(-m/m')=4ik'K'$.

**3.3. $\operatorname{am}(z; m)$ for $m>1$.** Denoting a parameter greater than 1 by $1/m$ where $0<m<1$, we have [5], [10], [15], [23]

$$K(1/m)=kK(m)+ikK'(m),\qquad K'(1/m)=kK'(m) \tag{3.7}$$

noting that, exactly as for (3.3), the sign on the right-hand side of this equation is arbitrary (Neville [19, pp. 103–107]). The expression for $\operatorname{am}(z; 1/m)$ from (2.9) becomes

$$\begin{aligned}\operatorname{am}\left(z; \frac{1}{m}\right)&=\sum_{n=-\infty}^{\infty}\operatorname{gd}\left[\frac{\pi K}{K'}\left(n+\frac{z}{2kK}\right)+in\pi\right]\\ &=\sum_{n=-\infty}^{\infty}(-1)^n\operatorname{gd}\left[\frac{\pi K}{K'}\left(n+\frac{z}{2kK}\right)\right].\end{aligned} \tag{3.8}$$

Then, with $|x/2kK|$ and $|y/kK'|<1$, we find, using (2.13),

$$\begin{aligned}\operatorname{am}&\left(z+2sK\left(\frac{1}{m}\right)+2itK'\left(\frac{1}{m}\right); \frac{1}{m}\right)\\ &=(-1)^s\operatorname{sgn}(x)\frac{1-(-1)^{s+t}}{2}\pi+(-1)^t\operatorname{am}\left(z; \frac{1}{m}\right).\end{aligned} \tag{3.9}$$

Thus we find that am $(z; 1/m)$ is a *doubly* periodic function, i.e., am $(z+2sK(1/m)+2itK'(1/m); 1/m) = \text{am}\,(z; 1/m)$ if *both* $s$ and $t$ are, independently, even integers. One of the periods, $4ikK'$, is strictly imaginary, while the other, $4kK$, is strictly real.

**4. Linear and quadratic transformations of the amplitude function.** This section will present the (linear) negative parameter, reciprocal parameter, and complementary parameter transformations as well as the (quadratic) Landen and Gauss transformations of the Jacobian amplitude function. Although only two linear transformations plus the Landen transformation are strictly necessary since the remaining linear and quadratic transformations can then be derived from these [8], [10], [20], all of the above-mentioned transformations are included here since they are the most familiar and widely used of the JEF transformations. The convention followed here for the nomenclature of the quadratic transformations is that defined by Carlson [6] for which the variable changes in the same/opposite manner as the parameter for the Gauss/Landen transformations. Concise yet general discussions of the transformation theory of elliptic functions may be found in the texts by Erdélyi et al. [10], Chandrasekharan [8], and Rauch and Lebowitz [20].

**4.1. The negative parameter (imaginary modulus) transformation.** The relationship between am $(z; m)$ and am $(z; -m/m')$ follows immediately from the identity given in (3.5), i.e.,

$$\text{(4.1)} \qquad \text{am}\left(k'z; -\frac{m}{m'}\right) = \frac{\pi}{2} - \text{am}\,(K-z; m), \qquad -\infty < m < 1$$

from which follow directly the JEF transformations $dn(k'z; -m/m') = nd(z; m)$, and so on. This transformation was first given by Jacobi in *Fundamenta Nova* [12, p. 90] and, apparently, subsequently forgotten. The amplitude function in the region $m<0$ is characteristically very similar to am $(z; m)$ with $0<m<1$. In each case, the function is singly periodic with an imaginary period $=4i\cdot\text{Re}\,[K'(m)]$ while, for $z$ strictly real, both am $(x; m)$ and am $(x; -m/m')$ are unbounded, monotonically increasing functions and so are invertible over the entire real axis. Finally, we note that limit am $(z; m) \equiv 0$ as $m \to -\infty$.

**4.2. The complementary parameter (imaginary argument) transformation.** Replacing $m$ with $m'$ in (2.9) gives immediately

$$\text{(4.2)} \qquad \text{am}\,(z; m') = \sum_{n=-\infty}^{\infty} \text{gd}\left[\frac{\pi K'}{K}\left(n + \frac{z}{2K'}\right)\right].$$

However, to relate am $(z; m)$ directly to am $(z; m')$, the familiar transformation for $dn(z; m')$ is rewritten as [5], [10], [23]

$$\text{(4.3)} \qquad dn(z; m') = \frac{d}{dz}\text{am}\,(z; m') = \frac{dn(iz; m)}{cn(iz; m)} = \frac{-i}{\cosh(i\Phi)}\frac{d\Phi}{dz} = \frac{d}{dz}\text{gd}\,(-i\Phi)$$

where $\phi = \text{am}\,(iz; m)$. Integrating from zero to $z$ yields the result

$$\text{(4.4)} \qquad \text{am}\,(z; m') = \text{gd}\,[-i\cdot\text{am}\,(iz; m)], \qquad 0 \leqq m \leqq 1$$

where $|\text{Re}\,(z)| < K'(m)$ and $|\text{Im}\,(z)| < K(m)$. Equations (3.1) and (3.2) may be used to extend the applicability of this result for arbitrary values of $z$. In certain respects, this transformation could be aptly subtitled the "circular-hyperbolic transformation" since it relates the "nearly circular" JEF to their "nearly hyperbolic" counterparts, i.e., the amplitude and elliptic functions with $m \approx 0$ to those with $m \approx 1$. In the extreme limit of $m = 0$, (4.4) gives am $(z; 1) = \text{gd}(z)$. Finally, note that the parameters $m$ and $m'$ are interchangeable in (4.4) so that am $(z; m) = \text{gd}[-i \cdot \text{am}(iz; m')]$.

**4.3. The descending Landen transformation.** Dealing for the moment with general, iterative subscripts, we define, for $0 \leqq m_i \leqq 1$, the transformations $m_{i+1} \leftrightarrow m_i$ as

$$m_{i+1} = f_-(m_i) = \left[\frac{1-k_i'}{1+k_i'}\right]^2, \qquad m_i = f_+(m_{i+1}) = \frac{4k_{i+1}}{(1+k_{i+1})^2} \tag{4.5}$$

such that $0 \leqq m_{i+1} \leqq m_i \leqq 1$ and $f_-[f_+(m)] = f_+[f_-(m)] = m$. The quarter periods for the two parameters connected by $f_-$ and $f_+$ are related as

$$K_{i+1} = \frac{1+k_i'}{2} K_i \quad \text{and} \quad K_{i+1}' = (1+k_i')K_i' \tag{4.6}$$

so that

$$\frac{K_{i+1}'}{K_{i+1}} = 2\frac{K_i'}{K_i} \quad \text{and} \quad q_{i+1} = q_i^2. \tag{4.7}$$

Setting $i = 0$ and using the notation $m_0 = m$, we can write the identities

$$\text{am}(z; m) = \sum_{n=-\infty}^{\infty} \text{gd}\left[\frac{\pi K_1}{K_1'}\left(2n + (1+k')\frac{z}{2K_1}\right)\right] \tag{4.8}$$

and

$$\text{am}(K - z; m) = \sum_{n=-\infty}^{\infty} \text{gd}\left[\frac{\pi K_1}{K_1'}\left(2n + 1 - (1+k')\frac{z}{2K_1}\right)\right]. \tag{4.9}$$

Combining these equations gives the descending Landen transformation for the amplitude function

$$\text{am}[(1+k')z; m_1] = \text{am}(z; m) - \text{am}(K - z; m) + \frac{\pi}{2}, \qquad 0 \leqq m < 1. \tag{4.10}$$

**4.4. The reciprocal parameter transformation.** Using the relations for the quarter periods, (3.7), along with (4.8) and (4.9) above, we have directly

$$\text{am}\left[(1-k')z; \frac{1}{m_1}\right] = \text{am}(z; m) + \text{am}(K - z; m) - \frac{\pi}{2}, \qquad 0 < m < 1. \tag{4.11}$$

Up to this point, all of the transformations given for am $(z; m)$ involve precisely those parameters that characterize the analogous JEF transformations. Although (4.11) is the correct, general form of the transformation relating the amplitude functions in the regions $0 < m < 1$ and $m > 1$, it does not directly relate $m$ to $1/m$ as its name suggests. To resolve this point, reference is made to (3.9) and the fact that, when and only when $m > 1$, is am $(z; m)$ a strictly oscillatory function with respect to *both* $K$ and $K'$. Hence, except at its branchpoints, the function am $(z; m > 1)$ is bounded throughout the entire complex plane, and we may properly represent it as an inverse of some combination of JEF. From the familiar identity for $dn(u+v; m)$ it follows that, for $|\text{Re}(u)/K|$, $|\text{Re}(v)/K| < 1$ and $|\text{Im}(u)/K'|$, $|\text{Im}(v)/K'| < 1$,

$$\text{am}(u+v; m) = \tan^{-1}[sc(u; m)\, dn(v; m)] + \tan^{-1}[dn(u; m) sc(v; m)]. \tag{4.12}$$

In particular, setting $u = kx$ and $v = iky$ leads to

$$\begin{aligned}\text{am}(kx + iky; m) = {} & \tan^{-1}[sc(kx; m) dc(ky; m')] \\ & + i \cdot \tanh^{-1}[dn(kx; m) sn(ky; m')].\end{aligned} \tag{4.13}$$

Transforming $m \to 1/m$ and rearranging terms within the brackets results in

$$\text{am}\left(kx+iky; \frac{1}{m}\right)=\sin^{-1}\left[\frac{k\cdot sn(x;m)}{[1-dn^2(x;m)sn^2(y;m')]^{1/2}}\right] + i\cdot\tanh^{-1}[k\cdot cn(x;m)sd(y;m')]. \tag{4.14}$$

In the strictest sense, relations such as these are incorrect whenever $1/m<1$ unless the values of the real and imaginary parts of the variable $z$ are specifically restricted as stated above or as in (3.1) and (3.2). In contrast, however, the identity of (4.14) is valid *as written* when $1/m>1$ for arbitrary $z$, the branchpoints excepted. Thus, the reciprocal parameter transformation for the amplitude function, (4.14), can be rewritten succinctly (although essentially symbolically when $z$ is complex) in the form

$$\text{am}\,(kz; 1/m)=\sin^{-1}[k\cdot sn(z;m)], \qquad 0<m\leqq 1 \tag{4.15}$$

noting that limit $\text{am}\,(z; m)\equiv 0$ as $m\to\infty$. In particular, note that, for $z$ real (or $y=4tK'$), the pragmatic identity $\text{am}\,(kx; 1/m)=\sin^{-1}[k\cdot sn(x;m)]$ gives the correct value of the amplitude function for all values of $x$.

The similarity between the reciprocal parameter transformation in the form of (4.11) and the descending Landen transformation, (4.10), is remarkable and is, in part, related to the fact that $f_+(m_1)=f_+(1/m_1)$. By combining these two equations and invoking the result in (4.1), it is possible to write a completely general identity that relates the amplitude functions in the three regions of real $m$ as follows:

$$\text{am}\,(z;m)=\text{am}\,(k'z; -m/m')+\text{am}\,((1-k')z; 1/m_1) \tag{4.16}$$

with $(-m/m')\leqq 0\leqq m\leqq 1\leqq(1/m_1)$.

**4.5. The ascending Landen transformation.** Adding (4.10) and (4.11) leads immediately to the ascending Landen transformation for the amplitude function as follows:

$$\begin{aligned}\text{am}\,(z;m)&=\tfrac{1}{2}\text{am}\,[(1+k')z; m_1]+\tfrac{1}{2}\text{am}\,[(1-k')z; 1/m_1]\\ &=\tfrac{1}{2}\text{am}\,[(1+k')z; m_1]+\tfrac{1}{2}\sin^{-1}[k_1 sn[(1+k')z; m_1]].\end{aligned} \tag{4.17}$$

Note that many texts refer simply to "the Landen transformation," invariably meaning the descending Landen transformation corresponding to (4.10) above. The relationships for the JEF corresponding to (4.10) are rational ones [5], [23], whereas those corresponding to the ascending Landen transformation, (4.17), are algebraic relations. This is one instance in which, apart from its conciseness, the form of the amplitude function transformation is simple and straightforward in comparison to its JEF counterpart.

**4.6. The ascending/descending Gauss transformations.** The ascending/descending Gauss transformations are derived by combining the complementary parameter transformation with the descending/ascending Landen transformations [8], [10], [20]. From (4.4) and (4.10), for $0<m<1$, follows the ascending Gauss transformation in the form

$$\text{gd}\,[i\cdot\text{am}\,[(1+k_1)z; m]]=\text{gd}\,[i\cdot\text{am}\,(z; m_1)]+\text{gd}\,[i\cdot\text{am}\,(z+iK_1'; m_1)]+\frac{\pi}{2} \tag{4.18}$$

while, from (4.4) and (4.17), the descending Gauss transformation is found to be

$$\begin{aligned}\text{gd}\,[i\cdot\text{am}\,(z;m_1)]&=\frac{1}{2}\text{gd}\,[i\cdot\text{am}\,(1+k_1)z; m]\\ &\quad+\frac{1}{2}\text{gd}\left[i\frac{\pi}{2}-i\cdot\text{am}\,[K-(1+k_1)z; m]\right].\end{aligned} \tag{4.19}$$

As with the Landen transformations, many texts that refer simply to "the Gauss transformation" invariably mean the ascending Gauss transformation for which the transformation formulae for the JEF are rational expressions [5], [23], unlike those corresponding to the descending Gauss transformation for which the JEF identities are algebraic.

**5. The method of the arithmetic-geometric mean and am $(x; m)$.** Although, in principle, the method of the arithmetic-geometric mean (AGM) (or the theta functions) could be employed for a complex variable (and, conceivably, for complex $m$ [9]), it is considerably more practical to calculate the real and imaginary parts of am $(z; m)$ and the JEF separately using identities such as (4.13). Thus, only strictly real variables $x$ will hereinafter be considered. In addition, this section will deal with the "classical" method of the AGM (e.g., King [14]) utilizing various trigonometric or hyperbolic recursion identities as opposed to purely algebraic versions [6], [21]. The former, while nominally less efficient computationally, offer the advantage of calculating the true value of am $(x; m)$ for arbitrarily large $|x|$, i.e., including the contribution $s\pi$ given in (3.2). Moreover, while the relations between the method of the AGM and the transformations presented here hold true whichever version is adopted, these relations are more explicit and thus more readily recognized in the "classical" case.

The method of the AGM begins by iteratively calculating, with $0 < m < 1$, the trio of numbers

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \quad b_{n+1} = (a_n b_n)^{1/2}, \quad c_{n+1} = \tfrac{1}{2}(a_n - b_n) \tag{5.1}$$

with starting values of $a_0 = 1$, $b_0 = k'$, and $c_0 = k$. The numbers $a_n$ and $b_n$ converge rapidly to a common limit $(=\pi/2K)$ while $c_n$, as a measure of the "error" with $c_n^2 = a_n^2 - b_n^2$, vanishes quadratically, i.e., $c_{n+1} = c_n^2/4a_{n+1}$. The calculation is stopped at the $N$th step where, to some prescribed degree of accuracy, $c_N$ is negligible. For the descending Landen version of the AGM, a sequence of angles $\phi_{N-1}, \phi_{N-2}, \cdots, \phi_0$ is then calculated sequentially using the recurrence relation

$$\sin(2\Phi_{n-1} - \Phi_n) = \frac{c_n}{a_n} \sin(\Phi_n) \quad \text{with } \Phi_N = 2^N a_N x \tag{5.2}$$

to which the amplitude function and the JEF are related as

$$\begin{aligned} &\text{am}(x; m) = \Phi_0, \qquad \text{am}(K - x; m) = \Phi_0 - \Phi_1 + \frac{\pi}{2}, \\ &sn(x; m) = \sin(\Phi_0), \qquad cn(x; m) = \cos(\Phi_0), \\ &dn(x; m) = \frac{cn(x; m)}{sn(K - x; m)} = \frac{\cos(\Phi_0)}{\cos(\Phi_0 - \Phi_1)}. \end{aligned} \tag{5.3}$$

Attention is drawn to the first two identities of (5.3) that, to the best of the author's knowledge, have not heretofore been given in any of the papers dealing with the AGM as a method of evaluating the JEF. Yet, given these two results, it follows that the evaluation of the JEF is, in fact, entirely incidental to this method, i.e., it is am $(x; m)$ and am $(K - x; m)$, which are the primary quantities found via the AGM from which the JEF are then calculated from simple trigonometric expressions.

To indeed establish the validity of the identities listed in (5.3), it is first noted that the $(a_n, b_n, c_n)$ scale described by (5.1) is exactly equivalent to sequential applications of the descending operator $f_-(m_i)$ of (4.5), i.e.,

$$
\begin{aligned}
k_0 &= k \\
k_1 &= (1-k_0')/(1+k_0') = f_-(m) = c_1/a_1 \\
k_2 &= (1-k_1')/(1+k_1') = f_-^2(m) = c_2/a_2 \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
k_{n+1} &= (1-k_n')/(1+k_n') = f_-^{n+1}(m) = c_{n+1}/a_{n+1}
\end{aligned}
\tag{5.4}
$$

and where $K(m) = (\pi/2)(1+k_1)(1+k_2)\cdots = (\pi/2a_N)$. With the identification of $k_n = c_n/a_n$, the recurrence relation (5.2) is immediately recognized as the ascending Landen transformation for the amplitude function, (4.17), so that the $\phi_n$ sequence is equivalent to

$$
\begin{aligned}
\Phi_0 &= \operatorname{am}(x;\, m) \\
\Phi_1 &= \operatorname{am}\left(\frac{2x}{(1+k_1)};\, m_1\right) \\
\Phi_2 &= \operatorname{am}\left(\frac{4x}{(1+k_1)(1+k_2)};\, m_2\right) \\
&\vdots \qquad\qquad \vdots \\
\Phi_n &= \operatorname{am}\left(\frac{2^n x}{(1+k_1)(1+k_2)\cdots(1+k_n)};\, m_n\right).
\end{aligned}
\tag{5.5}
$$

Note that, even though it is the *descending* Landen transformation that is the basis of this particular version of the AGM and transforms the variable $\phi_0 \to \phi_N$ as immediately above, it is the *ascending* Landen transformation for $\operatorname{am}(x;\, m)$ that is used in the actual calculations to transform $\phi_N \to \phi_0$. With the aid of (4.10), the sequence of amplitude functions, (5.5), may be re-expressed as

$$
\begin{aligned}
\Phi_n &= \operatorname{am}(x;\, m) + (1-\delta_{n0})\left[\frac{\pi}{2} - \operatorname{am}(K-x;\, m)\right] \\
&\quad + (1-\delta_{n0})(1-\delta_{n1}) \sum_{i=1}^{n-1} \sum_{j=1}^{2^{i-1}} \\
&\quad \cdot \left[\operatorname{am}\left[\frac{2j-1}{2^i}K + x;\, m\right] - \operatorname{am}\left[\frac{2j-1}{2^i}K - x;\, m\right]\right]
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
\Phi_n &= \frac{\pi}{2K}2^n x + \sum_{j=1}^{\infty} \frac{2q^{2^n j}}{j(1+q^{2^{n+1} j})} \sin\left(\frac{2^n j\pi x}{K}\right) \\
&= \frac{\pi}{2K}2^n x + \sum_{j=1}^{\infty} \frac{1}{j} \operatorname{sech}\left[\frac{2^n j\pi K'}{K}\right] \sin\left(\frac{2^n j\pi x}{K}\right).
\end{aligned}
\tag{5.7}
$$

Equation (5.6), in particular, establishes the identities given in (5.3), while (5.7) offers explicit testimony to the extraordinarily rapid convergence inherent to the AGM. This latter equation states that, for the $n$th step, the first-order deviation of the variable $\phi_n$ from linearity will go as $q$ raised to the $2^n$ power, i.e., given the relation of (4.7) for the nome, the "error" is reduced quadratically on each iteration.

The particular version of the AGM as just described with $b_0 = k'$ and the use of the recurrence relation (5.2) is referred to as the descending Landen version of the AGM. When we use $b_0 = k$ and (4.10) as the recurrence relation, the ascending Landen version of the AGM sequentially raises the parameter to unity where $\phi_N \approx \text{gd}$. Ascending and descending versions of the AGM based on Gauss transformations are also possible (Carlson [6]). The use of an ascending and a descending transformation, in particular, allows the numerical range of $m$ to be restricted to $0 < m \leqq \frac{1}{2}$. Whatever the particular version adopted, however, the calculation of the JEF from the final results, as for (5.3), proves to be incidental to the method in that the primary quantities that are calculated via the $\phi_n$ are the amplitude function am $(x; m)$ along with the "coam" function am $(K - x; m)$.

Calculation of am $(x; m)$ along with the JEF for parameter values outside the range of $0 < m < 1$ may be done as directly and efficiently as for the case of $0 < m < 1$ through the use of the transformation formulae (4.1) and (4.11). To calculate the amplitude function and the JEF for the case of a negative parameter $-M$ where $0 < M < \infty$, the AGM is calculated as above with a parameter $m = M/(1+M)$ and a variable $x/k'$ to give the results

$$\begin{gathered} \text{am}\,(x; -M) = \Phi_1 - \Phi_0, \qquad \text{am}\,(K(-M) - x; -M) = (\pi/2) - \Phi_0, \\ sn(x; -M) = \sin(\Phi_1 - \Phi_0), \qquad cn(x; -M) = \cos(\Phi_1 - \Phi_0), \\ dn(x; -M) = \cos(\Phi_1 - \Phi_0)/\cos(\Phi_0). \end{gathered} \tag{5.8}$$

To calculate the elliptic functions for the case of a parameter $M > 1$, the AGM as described above is calculated using a parameter $m = f_+(m_1)$, where $m_1 = 1/M$, and a variable $x/(1-k')$ to give the results

$$\begin{gathered} \text{am}\,(x; M) = 2\Phi_0 - \Phi_1, \qquad \text{am}\,(x/k_1; 1/M) = \Phi_1, \\ sn(x; M) = \sin(2\Phi_0 - \Phi_1), \qquad cn(x; M) = \cos(2\Phi_0 - \Phi_1), \\ dn(x; M) = \cos(\Phi_1). \end{gathered} \tag{5.9}$$

Note that, in the case of $M > 1$, it is not the coam function that is calculated along with am $(x; M)$ but rather am $(x/k_1; 1/M)$, which yields the value of $dn(x; M)$ directly. The directness of these algorithms, i.e., the calculation of the actual values of the amplitude function(s), may be contrasted with algorithms that calculate the JEF for parameters less than zero or greater than 1 as either rational or algebraic expressions involving values of JEF having $0 < m < 1$ (e.g., [4]).

**6. Concluding remarks.** The results presented in this paper have shown that the various linear and quadratic transformations of the JEF can be represented concisely by the corresponding transformation of the Jacobian amplitude function am $(z; m)$. The nature of the amplitude function for arbitrary, real $m$ has been shown to be markedly different according to whether $m < 1$ or $m > 1$, being a singly periodic function with a strictly imaginary period when $m < 1$ and $m \neq 0$ while, for $m > 1$, am $(z; m)$ is a doubly periodic function with both a real and an imaginary period. Finally, the method of the arithmetic-geometric mean has been shown to be principally a method for the calculation of the functions am $(x; m)$ and am $(K - x; m)$ directly from which then follow the values of the various JEF.

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